AH

National
Qualifications
2019

X747/77/11

Mathematics

THURSDAY, 2 MAY
9:00 AM – 12:00 NOON

Total marks — 100

Attempt ALL questions.

You may use a calculator.

Full credit will be given only to solutions that contain appropriate working.

State the units for your answer where appropriate.

Answers obtained by readings from scale drawings will not receive any credit.

Write your answers clearly in the answer booklet provided. In the answer booklet, you must clearly identify the question number you are attempting.

Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not, you may lose all the marks for this paper.

Standard derivatives	
$f(x)$	$f'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\ln x$	$\dfrac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\dfrac{1}{a}\tan(ax)+c$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)+c$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)+c$
$\dfrac{1}{x}$	$\ln|x|+c$
e^{ax}	$\dfrac{1}{a}e^{ax}+c$

Summations

(Arithmetic series) $\qquad S_n = \dfrac{1}{2}n\left[2a+(n-1)d\right]$

(Geometric series) $\qquad S_n = \dfrac{a\left(1-r^n\right)}{1-r}, \; r \neq 1$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \text{where} \quad \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \dots$$

De Moivre's theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Vector product

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ, about the origin, $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Total marks — 100

Attempt ALL questions

1. (a) Differentiate $f(x) = x^6 \cot 5x$. **2**

   (b) Given $y = \dfrac{2x^3 + 1}{x^3 - 4}$, find $\dfrac{dy}{dx}$. Simplify your answer. **3**

   (c) For $f(x) = \cos^{-1} 2x$ evaluate $f'\left(\dfrac{\sqrt{3}}{4}\right)$. **3**

2. Matrix A is defined by

$$A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & p & 2 \\ -1 & -2 & 5 \end{pmatrix}$$

   where $p \in \mathbb{R}$.

   (a) Given that the determinant of A is 3, find the value of p. **3**

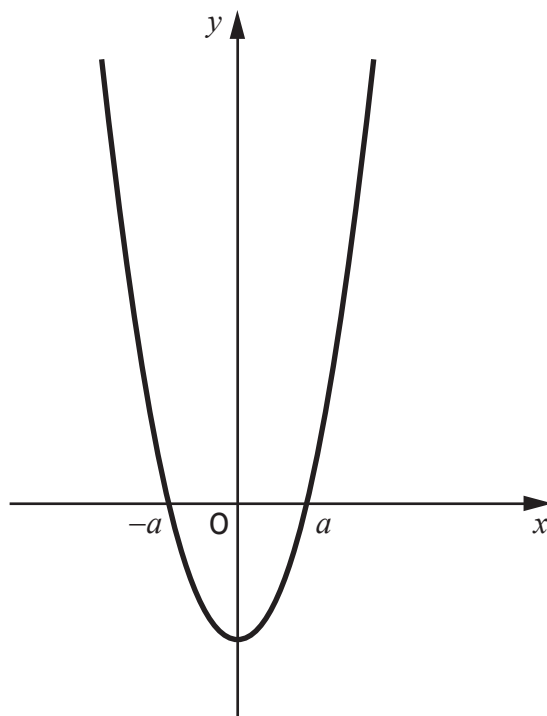   Matrix B is defined by

$$B = \begin{pmatrix} 0 & 1 \\ q & 3 \\ 4 & 0 \end{pmatrix}$$

   where $q \in \mathbb{R}$.

   (b) Find AB. **2**

   (c) Explain why AB does not have an inverse. **1**

3. The function $f(x)$ is defined by $f(x) = x^2 - a^2$. The graph of $y = f(x)$ is shown in the diagram.

(a) State whether $f(x)$ is odd, even or neither. Give a reason for your answer. **1**

(b) Sketch the graph of $y = |f(x)|$. **1**

4. (a) Express $\dfrac{3x^2 + x - 17}{x^2 - x - 12}$ in the form $p + \dfrac{qx + r}{x^2 - x - 12}$, where p, q and r are integers. **1**

(b) Hence express $\dfrac{3x^2 + x - 17}{x^2 - x - 12}$ with partial fractions. **3**

[Turn over

5. For $x = \ln(2t + 7)$ and $y = t^2$, $t > 0$, find

   (a) $\dfrac{dy}{dx}$ **2**

   (b) $\dfrac{d^2y}{dx^2}$. **2**

6. A spherical balloon of radius r cm, $r > 0$, deflates at a constant rate of $60\,\text{cm}^3\,\text{s}^{-1}$.

   Calculate the rate of change of the radius with respect to time when $r = 3$. **3**

   $$\left[\text{The volume of a sphere is given by } V = \frac{4}{3}\pi r^3. \right]$$

7. (a) Find an expression for $\displaystyle\sum_{r=1}^{n}(6r + 13)$ in terms of n. **1**

   (b) Hence, or otherwise, find $\displaystyle\sum_{r=p+1}^{20}(6r + 13)$. **2**

8. Find the particular solution of the differential equation

   $$\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 28y = 0$$

   given that $y = 0$ and $\dfrac{dy}{dx} = 9$, when $x = 0$. **5**

9. (a) Write down and simplify the general term in the binomial expansion of $\left(2x^2 - \dfrac{d}{x^3}\right)^7$, where d is a constant. **3**

(b) Given that the coefficient of $\dfrac{1}{x}$ is $-70\,000$, find the value of d. **2**

10. A curve is defined implicitly by the equation $x^2 + y^2 = xy + 12$.

(a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y. **3**

(b) There are two points where the tangent to the curve has equation $x = k$, $k \in \mathbb{R}$. Find the values of k. **2**

11. Let n be a positive integer.

(a) Find a counterexample to show that the following statement is false.

$n^2 + n + 1$ is always a prime number. **1**

(b) (i) Write down the contrapositive of:

If $n^2 - 2n + 7$ is even then n is odd. **1**

(ii) Use the contrapositive to prove that if $n^2 - 2n + 7$ is even then n is odd. **3**

12. Express 231_{11} in base 7. **3**

[Turn over

13. An electronic device contains a timer circuit that switches off when the voltage, V, reaches a set value.

The rate of change of the voltage is given by

$$\frac{dV}{dt} = k(12 - V),$$

where k is a constant, t is the time in seconds, and $0 \le V < 12$.

Given that $V = 2$ when $t = 0$, express V in terms of k and t. **5**

14. Prove by induction that

$$\sum_{r=1}^{n} r!r = (n+1)! - 1 \text{ for all positive integers } n.$$ **5**

15. The equations of two planes are given below.

$$\pi_1: \ 2x - 3y - z = 9$$
$$\pi_2: \ x + y - 3z = 2$$

(a) Verify that the line of intersection, L_1, of these two planes has parametric equations

$$x = 2\lambda + 3$$
$$y = \lambda - 1$$
$$z = \lambda$$ **2**

(b) Let π_3 be the plane with equation $-2x + 4y + 3z = 4$.

Calculate the acute angle between the line L_1 and the plane π_3. **3**

(c) L_2 is the line perpendicular to π_3 passing through P$(1, 3, -2)$.

Determine whether or not L_1 and L_2 intersect. **4**

16. (a) Use integration by parts to find the exact value of $\int_0^1 \left(x^2 - 2x + 1\right) e^{4x}\, dx$. **5**

(b) A solid is formed by rotating the curve with equation $y = 4(x - 1) e^{2x}$ between $x = 0$ and $x = 1$ through 2π radians about the x-axis.

Find the exact value of the volume of this solid. **3**

17. The first three terms of a sequence are given by

$$5x + 8,\ -2x + 1,\ x - 4$$

(a) When $x = 11$, show that the first three terms form the start of a geometric sequence, and state the value of the common ratio. **2**

(b) Given that the entire sequence is geometric for $x = 11$

  (i) state why the associated series has a sum to infinity **1**

  (ii) calculate this sum to infinity. **2**

(c) There is a second value for x that also gives a geometric sequence.
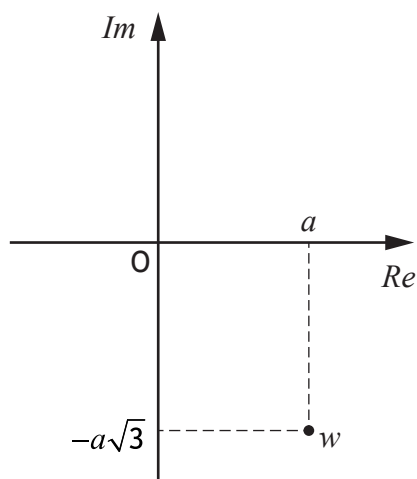
For this second sequence

  (i) show that $x^2 - 8x - 33 = 0$ **2**

  (ii) find the first three terms **2**

  (iii) state the value of S_{2n} and justify your answer. **1**

[Turn over for next question

18. The complex number w has been plotted on an Argand diagram, as shown below.

(a) Express w in

   (i) Cartesian form   **1**

   (ii) polar form.   **3**

(b) The complex number z_1 is a root of $z^3 = w$, where

$$z_1 = k\left(\cos\frac{\pi}{m} + i\sin\frac{\pi}{m}\right)$$

for integers k and m.

Given that $a = 4$,

   (i) use de Moivre's theorem to obtain the values of k and m, and   **4**

   (ii) find the remaining roots.   **2**

[END OF QUESTION PAPER]